Linear Vector Spaces, Orthogonality Principle, Cauchy-Schwarz Inequality, Matched Filter, Fourier Series and Fourier Transform Properties, and Poisson Sum Formulas

1 Linear Vector Spaces

Definition 1.1 (Linear Vector Space) *A* **linear vector space** $\mathcal{X}$ *is a set of elements called* **vectors** *together with two operations:*

1. **Addition.** $x \in \mathcal{X}$ and $y \in \mathcal{X} \Rightarrow x + y \in \mathcal{X}$.
2. **Scalar multiplication.** $x \in \mathcal{X}, \alpha \in \mathcal{C} \Rightarrow \alpha x \in \mathcal{X}$, *where* $\mathcal{C}$ *is the set of all complex numbers.*

These two operations satisfy the following properties:

1. **Commutative law of vector addition.** $x + y = y + x$
2. **Associative law of vector addition.** $(x + y) + z = x + (y + z)$
3. **Additive identity.** *There exists a* $\theta \in \mathcal{X}$ *such that* $x + \theta = x$.
4. $\alpha(x + y) = \alpha x + \alpha y$.
5. $(\alpha + \beta)x = \alpha x + \beta x$.
6. $(\alpha\beta)x = \alpha(\beta x)$.
7. $0x = \theta$.
8. $1x = x$.

Definition 1.2 (Norm) *The quantity* $\|x\|$ *is a* **norm** *on a linear vector space* $\mathcal{X}$ *if:*

1. $\|x\| \geq 0$ *for all* $x \in \mathcal{X}$, *with* $\|x\| = 0$ *iff (if and only if)* $x = \theta$;
2. **Triangle inequality:** $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in \mathcal{X}$; *and*
3. $\|\alpha x\| = |\alpha|\|x\|$.

Definition 1.3 (Inner product) *The quantity* $\langle x, y\rangle$ *is an* **inner product** *defined on* $\mathcal{X} \times \mathcal{X}$ *(meaning that* $x \in \mathcal{X}$ *and* $y \in \mathcal{X}$*), where* $\mathcal{X}$ *is a linear vector space, if* $\langle x, y\rangle \in \mathcal{C}$ *and the following properties are all satisfied:*

1. $\langle x, y\rangle = \langle y, x\rangle^*$.
2. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.
3. $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$.
4. $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ *iff* $x = \theta$.

Note: The quantity $\sqrt{\langle x, x\rangle}$ is a valid norm. We will use $\|x\|^2 = \langle x, x\rangle$ extensively.

Definition 1.4 (Orthogonal vectors) *Two elements x and y of a linear vector space $\mathcal{X}$ are said to be* **orthogonal** *(or perpendicular) if $\langle x, y\rangle = 0$, in which case we often write $x \perp y$.*

Note: The set of all energy signals is a linear vector space. For two continuous-time energy signals $x(\cdot)$ and $y(\cdot)$, we use the inner product

$$\langle x, y\rangle = \int_{-\infty}^{\infty} x(t)y^*(t)\,dt.$$

For two discrete-time energy signals $x[\cdot]$ and $y[\cdot]$, we use the inner product

$$\langle x, y\rangle = \sum_{n=-\infty}^{\infty} x[n]y^*[n].$$

Also note that $\langle x, x\rangle = \|x\|^2 = E_x$ (the **energy** in $x(\cdot)$) is a valid norm.

Note: The set of all power signals is a linear vector space. For two continuous-time power signals $x(\cdot)$ and $y(\cdot)$, we use the inner product

$$\langle x, y\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)y^*(t)\,dt.$$

For two discrete-time power signals $x[\cdot]$ and $y[\cdot]$, we use the inner product

$$\langle x, y\rangle = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{-N}^{N} x[n]y^*[n].$$

Also note that $\langle x, x\rangle = \|x\|^2 = P_x$ (the **power** in $x(\cdot)$) is a valid norm.

2 Orthogonality Principle

Theorem 2.1 (Orthogonality Principle) *Let $\mathcal{X}$ be a linear vector space, $x \in \mathcal{X}$ and let $\{\phi_n\}_{n=1}^N$ be a set of* **basis vectors** *in $\mathcal{X}$. We want to estimate x with $\hat{x}_N$, where*

$$\hat{x}_N = \sum_{n=1}^{N} c_n \phi_n$$

is a linear combination of the N basis vectors. We want to find the N constants $\{c_n\}_{n=1}^N$ to minimize the norm-squared error between x and $\hat{x}_N$; i.e., find $\{c_n\}_{n=1}^N$ to minimize

$$\mathcal{E}_N = \|x - \hat{x}_N\|^2 = \langle x - \hat{x}_N, x - \hat{x}_N\rangle.$$

The best choice satisfies the **orthogonality principle: The error must be orthogonal to the data used in the estimate**, *or*

$$\text{error} = x - \hat{x}_N \perp \phi_\ell, \qquad \ell = 1, 2, \ldots, N.$$

Consequently,

$$\langle x - \hat{x}_N, \phi_\ell \rangle = 0, \qquad \ell = 1, 2, \ldots, N,$$

or

$$\langle \hat{x}_N, \phi_\ell \rangle = \langle x, \phi_\ell \rangle, \qquad \ell = 1, 2, \ldots, N.$$

Substituting for $\hat{x}_N$, we find that the c's must satisfy

$$\sum_{n=1}^{N} c_n \langle \phi_n, \phi_\ell \rangle = \langle x, \phi_\ell \rangle, \qquad \ell = 1, 2, \ldots, N.$$

The resulting minimum value of $\mathcal{E}_N$ is

$$\mathcal{E}_{N,\min} = \|x\|^2 - \sum_{n=1}^{N} c_n \langle \phi_n, x \rangle.$$

If $\{\phi_n\}_{n=1}^{N}$ is an orthogonal set of basis vectors (meaning that $\langle \phi_\ell, \phi_n \rangle = 0$ for $\ell \neq n$), then

$$c_n = \frac{\langle x, \phi_n \rangle}{\|\phi_n\|^2}.$$

Proof. Let

$$x_N^{\perp} = \sum_{n=1}^{N} c_n^{\perp} \phi_n$$

satisfy

$$\langle x - x_N^{\perp}, \phi_\ell \rangle = 0 \qquad \text{for } \ell = 1, 2, \ldots, N.$$

Using the "give and take trick" we find

$$\begin{aligned}
\mathcal{E}_N &= \|x - \hat{x}_N\|^2 \\
&= \|(x - x_N^{\perp}) + (x_N^{\perp} - \hat{x}_N)\|^2 \\
&= \|x - x_N^{\perp}\|^2 + \langle x - x_N^{\perp}, x_N^{\perp} - \hat{x}_N \rangle + \langle x_N^{\perp} - \hat{x}_N, x - x_N^{\perp} \rangle + \|x_N^{\perp} - \hat{x}_N\|^2 \\
&= \|x - x_N^{\perp}\|^2 + 2\Re e(\langle x - x_N^{\perp}, x_N^{\perp} - \hat{x}_N \rangle) + \|x_N^{\perp} - \hat{x}_N\|^2.
\end{aligned}$$

Now, since $x_N^{\perp}$ and $\hat{x}_N$ are both linear combinations of $\{\phi_n\}_{n=1}^{N}$, and since $\langle x - x_N^{\perp}, \phi_\ell \rangle = 0$, we have

$$\langle x - x_N^{\perp}, x_N^{\perp} - \hat{x}_N \rangle = 0.$$

Consequently,

$$\begin{aligned}
\mathcal{E}_N &= \|x - \hat{x}_N\|^2 \\
&= \|x - x_N^{\perp}\|^2 + \|x_N^{\perp} - \hat{x}_N\|^2 \\
&\geq \|x - x_N^{\perp}\|^2,
\end{aligned}$$

with $\mathcal{E}_N = \|x - x_N^{\perp}\|^2 = \mathcal{E}_{N,\min}$ iff $\hat{x}_N = x_N^{\perp}$.

Note that

$$\begin{aligned}
\mathcal{E}_{N,\min} &= \langle x - x_N^{\perp}, x - x_N^{\perp} \rangle \\
&= \langle x - x_N^{\perp}, x \rangle - \langle x - x_N^{\perp}, x_N^{\perp} \rangle \\
&= \langle x - x_N^{\perp}, x \rangle \\
&= \|x\|^2 - \sum_{n=1}^{N} c_n^{\perp} \langle \phi_n, x \rangle
\end{aligned}$$

3 Cauchy-Schwarz Inequality

Theorem 3.1 (Cauchy-Schwarz Inequality) *Let $\mathcal{X}$ be a linear vector space, $x \in \mathcal{X}$, $y \in \mathcal{X}$, and let $\langle x, y \rangle$ be an inner product on $\mathcal{X} \times \mathcal{X}$, with norm defined by $\|x\|^2 = \langle x, x \rangle$. Then*

$$|\langle x, y \rangle| \leq \|x\| \|y\|, \tag{1}$$

with equality iff either $x = cy$ for any constant c or $y = \theta$.

Proof. First, note that if $x = cy$, then

$$|\langle x, y \rangle| = |c\langle y, y \rangle| = |c| \|y\|^2 = |c| \|y\| \|y\| = \|x\| \|y\|.$$

Further, if $y = \theta$, then $|\langle x, y \rangle| = 0 = \|x\| \|y\|$.

Let $\hat{x} = \lambda y$ and $\mathcal{E} = \|x - \hat{x}\|^2$. From the orthogonality principle, we know that the λ which minimizes $\mathcal{E}$ must satisfy

$$x - \hat{x} \perp y$$

so that $\langle x - \hat{x}, y \rangle = 0$, or $\langle \hat{x}, y \rangle = \langle x, y \rangle$. We conclude that the minimizing value of λ is

$$\lambda = \frac{\langle x, y \rangle}{\|y\|^2},$$

and the resulting minimum value of $\mathcal{E}$ is

$$\mathcal{E}_{\min} = \langle x - \hat{x}, x \rangle = \|x\|^2 - \lambda \langle y, x \rangle = \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2}$$

Consequently, since $\mathcal{E}_{\min} \geq 0$,

$$|\langle x, y \rangle|^2 \leq \|x\|^2 \|y\|^2.$$

4 Matched Filter

The matched filter is frequently used in many communications systems, including virtually all digital communication systems and radar/sonar systems. The basic premise is to design a system to maximize the output signal-to-noise ratio at a particular time. Here, we introduce the main ingredient.

Consider an LTI system with input a known energy signal $x(\cdot)$, impulse response $h(\cdot)$, and output

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau.$$

We want to design the LTI system (i.e., find $h(\cdot)$) to maximize $|y(t_o)|$:

$$|y(t_o)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t_o-\tau)\, d\tau \right| = \left| \int_{-\infty}^{\infty} h(\tau)g^*(\tau)\, d\tau \right| = |\langle h, g\rangle| \leq \|h\|\|g\|,$$

where we have applied the Cauchy-Schwarz inequality with $g^*(\tau) = x(t_o - \tau)$. The upper bound is achieved when

$$h(\tau) = cg(\tau) = cx^*(t_o - \tau),$$

which yields

$$|y(t_o)| = |c|\|x\|^2 = |c|E_x.$$

5 Complex Fourier Series

A very important application of the Orthogonality Principle is the complex Fourier series expansion. Let

$$\hat{x}_N(t) = \sum_{n=-N}^{N} X_n e^{j2\pi n f_o t}.$$

We want to find $\{X_n\}_{n=-N}^{N}$ to minimize

$$\mathcal{E}_N = \frac{1}{T_o} \int_{t_o}^{t_o+T_o} |x(t) - \hat{x}_N(t)|^2\, dt,$$

where $T_o = 1/f_o$. Applying the Orthogonality Principle we identify $\phi_n = \phi_n(t) = e^{j2\pi n f_o t}$ and

$$\mathcal{E}_N = \|x - \hat{x}_N\|^2 = \frac{1}{T_o} \int_{t_o}^{t_o+T_o} |x(t) - \hat{x}_N(t)|^2\, dt,$$

so that our inner product is

$$\langle x, y\rangle = \frac{1}{T_o} \int_{t_o}^{t_o+T_o} x(t)y^*(t)\, dt,$$

on the linear vector space of all functions which are square integrable on the interval $[t_o, t_o + T_o)$.

Note that our basis functions are **orthonormal** (orthogonal and unit-norm) since they satisfy

$$\langle \phi_\ell, \phi_n\rangle = \frac{1}{T_o} \int_{t_o}^{t_o+T_o} e^{j2\pi \ell f_o t} e^{-j2\pi n f_o t}\, dt = \begin{cases} 1, & \text{if } \ell = n \\ 0, & \text{if } \ell \neq n. \end{cases}$$

We conclude that

$$X_n = \langle x, \phi_n\rangle = \frac{1}{T_o} \int_{t_o}^{t_o+T_o} x(t) e^{-j2\pi n f_o t}\, dt.$$

The minimum value of the mean-square error is $\mathcal{E}_{N,\min} = \|x\|^2 - \sum_{n=-N}^{N} |X_n|^2$.

It can be shown (not easily!) that $\mathcal{E}_{N,\min} \to 0$ as $N \to \infty$, so that

$$\|x\|^2 = \frac{1}{T_o} \int_{t_o}^{t_o+T_o} |x(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |X_n|^2.$$

6 Fourier Series Properties

Consider periodic signals, $x(\cdot)$, $y(\cdot)$, $g(\cdot)$, with common period $T_o = 1/f_o$. The complex Fourier series for $x(t)$ (also called the **synthesis equation**)is

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_o t}.$$

The nth Fourier series coefficient for $x(t)$ can be found (at least conceptually) using the **analysis equation**:

$$X_n = \frac{1}{T_o} \int_{t_o}^{t_o+T_o} x(t) e^{-j2\pi n f_o t} \, dt,$$

where t_o is any real number convenient for the indicated integration.

By using the properties of Fourier series below, a **boring, tedious** integration can very often be avoided.

1. **Linearity.** If $g(t) = \alpha x(t) + \beta y(t)$, where α and β are complex constants, then $G_n = \alpha X_n + \beta Y_n$.

   (a) **DC Shift.** If $g(t) = x(t) + \beta$, then $G_n = X_n + \beta \delta_n$, where δ_n is the Kronecker delta function ($\delta_n = 1$ if $n = 0$, $\delta_n = 0$ if $n \neq 0$).

2. **Complex Conjugation.** If $g(t) = x^*(t)$ then $G_n = X^*_{-n}$.

   (a) If $x(t) = x^*(t)$ ($x(\cdot)$ is a real signal), then $X_n = X^*_{-n}$ ($\{X_n\}$ is an Hermitian discrete-frequency signal).

   (b) If $x(t) = -x^*(t)$ ($x(\cdot)$ is an imaginary signal), then $X_n = -X^*_{-n}$ ($\{X_n\}$ is skew Hermitian).

   (c) If $g(t) = \Re e(x(t)) = \frac{1}{2}x(t) + \frac{1}{2}x^*(t)$, then $G_n = \frac{1}{2}X_n + \frac{1}{2}X^*_{-n}$.

   (d) If $g(t) = \Im m(x(t)) = \frac{1}{j2}x(t) - \frac{1}{j2}x^*(t)$, then $G_n = \frac{1}{j2}X_n - \frac{1}{j2}X^*_{-n}$.

3. **Time reversal.** If $g(t) = x(-t)$ then $G_n = X_{-n}$.

   (a) If $x(t) = x(-t)$ ($x(\cdot)$ is an even signal), then $X_n = X_{-n}$ ($\{X_n\}$ is an even discrete-frequency signal).

   (b) If $x(t) = -x(-t)$ ($x(\cdot)$ is an odd signal), then $X_n = -X_{-n}$ ($\{X_n\}$ is an odd discrete-frequency signal).

(c) If $g(t) = x_e(t) = \frac{1}{2}x(t) + \frac{1}{2}x(-t)$, then $G_n = \frac{1}{2}X_n + \frac{1}{2}X_{-n}$.

(d) If $g(t) = x_o(t) = \frac{1}{2}x(t) - \frac{1}{2}x(-t)$, then $G_n = \frac{1}{2}X_n - \frac{1}{2}X_{-n}$.

4. **Complex conjugation and time-reversal.** If $g(t) = x^*(-t)$ then $G_n = X_n^*$.

   (a) If $x(t) = x^*(-t)$ ($x(\cdot)$ is an Hermitian signal), then $X_n = X_n^*$ ($\{X_n\}$ is a real discrete-frequency signal).

   (b) If $x(t) = -x^*(-t)$ ($x(\cdot)$ is a skew Hermitian signal), then $X_n = -X_n^*$ ($\{X_n\}$ is an imaginary discrete-frequency signal).

   (c) If $g(t) = x_h(t) = \frac{1}{2}x(t) + \frac{1}{2}x^*(-t)$, then $G_n = \frac{1}{2}X_n + \frac{1}{2}X_n^* = \Re e(X_n)$.

   (d) If $g(t) = x_{sh}(t) = \frac{1}{2}x(t) - \frac{1}{2}x^*(-t)$, then $G_n = \frac{1}{2}X_n - \frac{1}{2}X_n^* = j\Im m(X_n)$.

5. **Time Shift.** If $g(t) = x(t-\tau)$, then $G_n = X_n e^{-j2\pi n f_o \tau}$.

6. **Time Differentiation.** If $g(t) = Dx(t) = \dfrac{d}{dt}x(t)$, then $G_n = j2\pi n f_o X_n$.

7. **Cyclic** or **Periodic Convolution.** If $y(t) = \dfrac{1}{T_o}\displaystyle\int_0^{T_o} h(\tau)x(t-\tau)\,d\tau = \sum_{n=-\infty}^{\infty} Y_n e^{j2\pi n f_o t}$, then $Y_n = H_n X_n$.

8. **Cyclic** or **Periodic Correlation.** If

$$\rho_{x,y}(\tau) = g(\tau) = \frac{1}{T_o}\int_0^{T_o} x(t+\tau)y^*(t)\,dt = \sum_{n=-\infty}^{\infty} G_n e^{j2\pi n f_o \tau},$$

   then $G_n = X_n Y_n^*$.

   (a) **Parseval's Formula**: $\rho_{x,y}(0) = \dfrac{1}{T_o}\displaystyle\int_0^{T_o} x(t)y^*(t)\,dt = \sum_{n=-\infty}^{\infty} X_n Y_n^*$.

   (b) **Parseval's Formula**: $P_x = \rho_{x,x}(0) = \dfrac{1}{T_o}\displaystyle\int_0^{T_o} |x(t)|^2\,dt = \sum_{n=-\infty}^{\infty} |X_n|^2$.

9. **Inverse Discrete-time Fourier Transform.** The DTFT (Discrete-time Fourier Transform) of the discrete-time signal $\{x[n]\}$ is

$$g(f) = X(e^{j2\pi fT}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fnT} = \sum_{n=-\infty}^{\infty} x[n]e^{j2\pi fnT}.$$

   We recognize the righthand side as the complex Fourier series expansion of the function $g(f)$ which is periodic in f with period $1/T$. The nth Fourier series coefficient for $g(f)$ is $G_n = x[-n]$, with

$$G_{-n} = x[n] = T\int_{f_o}^{f_o+\frac{1}{T}} X(e^{j2\pi fT})e^{j2\pi fnT}\,df$$

   for any (real) choice of f_o. This equation is known as the inverse discrete-time Fourier transform (IDTFT).

10. **Inverse Z-Transform.** The Z-transform of the discrete-time signal $\{x[n]\}$ is

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \quad r_1 < |z| < r_2.$$

With $z = re^{j2\pi fT}$, $r_1 < |z| < r_2$, let

$$g(f) = X(z) = X(re^{j2\pi fT}) = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j2\pi fnT}.$$

Then $G_{-n} = x[n]r^{-n}$ so that $x_n = r^n G_{-n}$, or

$$x[n] = r^n T \int_0^{1/T} X(e^{j2\pi fnT})e^{j2\pi fnT}\, df = \frac{1}{j2\pi}\oint_{|z|=r} X(z)z^{n-1}\, dz,$$

where we have used $dz = j2\pi Tre^{j2\pi fT}df = j2\pi Tz\, df$. This equation is known as the inverse Z-transform integral. It can very often be easily evaluated using the residue theorem of analytic function theory (which is unfortunately out of the scope of this course).

7 Fourier Transform Properties

Consider energy signals, $x(\cdot)$, $y(\cdot)$, $g(\cdot)$, etc. The Fourier transform of $x(t)$ is given by the **analysis equation**

$$X(j\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt.$$

The corresponding inverse Fourier transform of $X(j\omega)$ is given by the **synthesis equation**

$$x(t) = \mathcal{F}^{-1}\{X(j\omega)\} = \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,\frac{d\omega}{2\pi}.$$

These integrations can very often be avoided by applying the Fourier transform properties below.

1. **Linearity.** If $g(t) = \alpha x(t) + \beta y(t)$, where α and β are complex constants, then $G(j\omega) = \alpha X(j\omega) + \beta Y(j\omega)$.

   (a) **DC Shift.** If $g(t) = x(t) + \beta$, then $G(j\omega) = X(j\omega) + \beta 2\pi\delta(\omega)$, where $\delta(\omega)$ is the Dirac delta function.

2. **Complex Conjugation.** If $g(t) = x^*(t)$ then $G(j\omega) = X^*(-j\omega)$.

   (a) If $x(t) = x^*(t)$ ($x(\cdot)$ is a real signal), then $X(j\omega) = X^*(-j\omega)$ ($X(\cdot)$ is an Hermitian frequency-domain signal).

   (b) If $x(t) = -x^*(t)$ ($x(\cdot)$ is an imaginary signal), then $X(j\omega) = -X^*(-j\omega)$ ($X(\cdot)$ is skew Hermitian).

   (c) If $g(t) = \Re e(x(t)) = \frac{1}{2}x(t) + \frac{1}{2}x^*(t)$, then $G(j\omega) = \frac{1}{2}X(j\omega) + \frac{1}{2}X^*(-j\omega)$.

   (d) If $g(t) = \Im m(x(t)) = \frac{1}{j2}x(t) - \frac{1}{j2}x^*(t)$, then $G(j\omega) = \frac{1}{j2}X(j\omega) - \frac{1}{j2}X^*(-j\omega)$.

3. **Time reversal.** If $g(t) = x(-t)$ then $G(j\omega) = X(-j\omega)$.
   (a) If $x(t) = x(-t)$ ($x(\cdot)$ is an even signal), then $X(j\omega) = X(-j\omega)$ ($X(\cdot)$ is even).
   (b) If $x(t) = -x(-t)$ ($x(\cdot)$ is an odd signal), then $X(j\omega) = -X(-j\omega)$ ($X(\cdot)$ is odd).
   (c) If $g(t) = x_e(t) = \frac{1}{2}x(t) + \frac{1}{2}x(-t)$, then $G(j\omega) = \frac{1}{2}X(j\omega) + \frac{1}{2}X(-j\omega)$.
   (d) If $g(t) = x_o(t) = \frac{1}{2}x(t) - \frac{1}{2}x(-t)$, then $G(j\omega) = \frac{1}{2}X(j\omega) - \frac{1}{2}X(-j\omega)$.

4. **Complex conjugation and time-reversal.** If $g(t) = x^*(-t)$ then $G(j\omega) = X^*(j\omega)$.
   (a) If $x(t) = x^*(-t)$ ($x(\cdot)$ is an Hermitian signal), then $X(j\omega) = X^*(j\omega)$ ($X(\cdot)$ is a real frequency-domain signal).
   (b) If $x(t) = -x^*(-t)$ ($x(\cdot)$ is a skew Hermitian signal), then $X(j\omega) = -X^*(j\omega)$ ($X(\cdot)$ is an imaginary frequency-domain signal).
   (c) If $g(t) = x_h(t) = \frac{1}{2}x(t) + \frac{1}{2}x^*(-t)$, then $G(j\omega) = \frac{1}{2}X(j\omega) + \frac{1}{2}X^*(j\omega) = \Re e(X())$.
   (d) If $g(t) = x_{sh}(t) = \frac{1}{2}x(t) - \frac{1}{2}x^*(-t)$, then $G(j\omega) = \frac{1}{2}X(j\omega) - \frac{1}{2}X^*(j\omega) = j\Im m(X(j\omega))$.

5. **Time Shift.** If $g(t) = x(t-\tau)$, then $G(j\omega) = X(j\omega)e^{-j\omega\tau}$.

6. **Frequency Shift.** If $G(j\omega) = X(j(\omega - \omega_o))$, then $g(t) = x(t)e^{j\omega_o t}$.

7. **Time Differentiation.** If $g(t) = Dx(t) = \dfrac{d}{dt}x(t)$, then $G(j\omega) = j\omega X(j\omega)$.

8. **Frequency Differentiation.** If $G(j\omega) = \dfrac{d}{d\omega}X(j\omega)$, then $g(t) = -jtx(t)$.

9. **Time-domain Convolution.** If
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau = \int_{-\infty}^{\infty} Y(j\omega)e^{j\omega t}\,\frac{d\omega}{2\pi},$$
then $Y(j\omega) = H(j\omega)X(j\omega)$.

10. **Frequency-domain Convolution.** If
$$Y(j\omega) = \int_{-\infty}^{\infty} H(j\lambda)X(j(\omega-\lambda))\,\frac{d\lambda}{2\pi} = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}\,dt,$$
then $y(t) = h(t)x(t)$.

11. **Correlation.** If
$$\rho_{x,y}(\tau) = g(\tau) = \int_{-\infty}^{\infty} x(t+\tau)y^*(t)\,dt = \int_{-\infty}^{\infty} G(j\omega)e^{j\omega\tau}\,\frac{d\omega}{2\pi},$$
then $G(j\omega) = X(j\omega)Y^*(j\omega)$.

(a) **Parseval's Formula.**

$$\rho_{x,y}(0) = \int_{-\infty}^{\infty} x(t)y^*(t)\,dt = \int_{-\infty}^{\infty} X(j\omega)Y^*(j\omega)\,\frac{d\omega}{2\pi}$$

(b) **Parseval's Formula.**

$$E_x = \rho_{x,x}(0) = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |X(j\omega)|^2\,\frac{d\omega}{2\pi}$$

12. **Time Scaling.** If $g(t) = x(at)$, then $G(j\omega) = \frac{1}{|a|}X(j\frac{\omega}{a})$.

13. **Duality.** If $g(t) = X(jt)$, then $G(j\omega) = 2\pi x(-\omega)$.

8 Poisson Sum Formulas

The Poisson sum formulas give us extremely useful relationships between sampling in one domain and a periodic function in the other domain. They are simply derived using both Fourier series and Fourier transforms.

8.1 PSF1

Let $x(\cdot)$ be an energy signal with Fourier transform $X(j\omega)$. Define the periodic function

$$\tilde{x}_P(t) = \sum_{k=-\infty}^{\infty} x(t - kP).$$

Since $\tilde{x}_P(t)$ is periodic in t with period P, we may expand it in a complex Fourier series ($f_o = 1/P$)

$$\tilde{x}_P(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_o t},$$

where ($t' = t - kP$)

$$\begin{aligned}
c_n &= \frac{1}{P}\int_0^P \tilde{x}_P(t)e^{-j2\pi n f_o t}\,dt \\
&= \frac{1}{P}\sum_{k=-\infty}^{\infty}\int_0^P x(t-kP)e^{-j2\pi n f_o t}\,dt \\
&= \frac{1}{P}\sum_{k=-\infty}^{\infty}\int_{-kP}^{P-kP} x(t')e^{-j2\pi n f_o t'}e^{-j2\pi n f_o kP}\,dt' \\
&= \frac{1}{P}\int_{-\infty}^{\infty} x(t')e^{-j2\pi n f_o t'}\,dt' \\
&= \frac{1}{P}X(jn\omega_o).
\end{aligned}$$

We thus have our first Poisson sum formula (PSF1):

$$\tilde{x}_P(t) = \sum_{k=-\infty}^{\infty} x(t-kP) = \sum_{n=-\infty}^{\infty} \frac{1}{P}X(jn\frac{2\pi}{P})e^{jn\frac{2\pi}{P}t}. \tag{2}$$

8.2 PSF2

Let $x(\cdot)$ be an energy signal with Fourier transform $X(j\omega)$. Define the periodic function

$$\tilde{X}_W(j\omega) = \sum_{k=-\infty}^{\infty} X(j(\omega - kW)).$$

Since $\tilde{X}_W(f)$ is periodic in ω with period W, we may expand it in a complex Fourier series ($T = 2\pi/W$)

$$\tilde{X}_W(j\omega) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega T},$$

where ($\omega' = \omega - kW$)

$$\begin{aligned} c_n &= \frac{1}{W}\int_0^W \tilde{X}_W(j\omega)e^{-jn\omega T}\,d\omega \\ &= \frac{1}{W}\sum_{k=-\infty}^{\infty}\int_0^W X(j(\omega - kW))e^{-jn\omega T}\,d\omega \\ &= \frac{2\pi}{W}\sum_{k=-\infty}^{\infty}\int_{-kW}^{W-kW} X(j\omega')e^{-jn\omega' T}e^{-jknWT}\,\frac{d\omega'}{2\pi} \\ &= T\int_{-\infty}^{\infty} X(j\omega')e^{-jn\omega' T}\,\frac{d\omega'}{2\pi} \\ &= Tx(-nT). \end{aligned}$$

We thus have our second Poisson sum formula (PSF2):

$$\tilde{X}_W(j\omega) = \sum_{k=-\infty}^{\infty} X(j(\omega - kW)) = \sum_{n=-\infty}^{\infty} Tx(nT)e^{-jn\omega T}. \tag{3}$$

(the end)